ON THE CONSTRUCTION OF A THEORY FOR AN IDEALLY PLASTIC BODY

(K POSTROENIIU TEORII IDEAL'NO PLASTICHESKOGO TELA)

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The starting points of a theory for an ideal rigid plastic body are the determination of the limiting surface and of the law of flow. If one may say concerning the latter that it has already been considered in a sufficiently general way different opinions exist concerning the plasticity condition.

Ivlev [1] was the first to consider the question of a possible choice of a plasticity condition, employing extremal principles. Upon investigating all possible limiting surfaces for the case of an ideal rigid plastic body, Ivlev showed that the Tresca plasticity condition was characterized by minimum work of the stresses for given incremental strains. This circumstance has as a basis the fact that the stress corresponding to the beginning of flow in tension (or an equal value in compression) occurs experimentally at a unique value. Other possible cases for the given initial experimental point were not considered in the above paper, for in the construction of an isotropic theory the choice of the initial point must not have an effect upon the results. It is shown that if any other point is taken as the given initial point, the Tresca plasticity condition loses its extremal property. The first section of this paper considers the question of the choice of a flow condition for the case of given limiting stress in tension, when use is made of two extremal principles. The general case of an arbitrary given point is investigated in the second section.

1. We assume that from experiments in tension or compression a value of stress is given for the beginning of flow (the same value for compression as for tension). Then, from the principle of symmetry and nonconcavity of the flow surface, it follows that this value must lie within the limits of two hexagonal prisms. The proof was given in the abovementioned paper and we shall not repeat it here.

Figure 1 shows the projections of the principal axes on the plane

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{const} \tag{1.1}$$

in the space of the principal stresses σ_1, σ_2, σ_3 which are perpendicular to these prisms. The two hexagons represent the intersections of the prisms with this plane. The points A_1 are given. The inside hexagon represents the Tresca plasticity condition. We call the outside hexagon the K-surface, for brevity.

The work of the stresses for incremental strains is determined from the following formula

$$dW = \sigma de \tag{1.2}$$

We investigate the minimum of dW for given incremental strains. We suppose that the modulus of the vector $d\mathbf{e}$ remains the same

$$(de_1)^2 + (de_2)^2 + (de_3)^2 = (da)^2 \tag{1.3}$$

for all cases. We now study the change in dW under the condition (1.3).

We prove that for all plasticity conditions the vector $d\mathbf{e}$ will take such a direction that the dissipation of energy dW will be determined by the equation

$$dW = 2K\sqrt{2/3}\, da \tag{1.4}$$

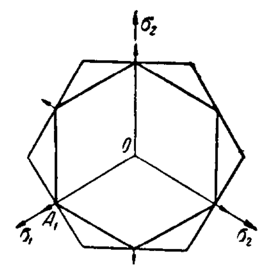

FIG. 1.

Either the curve is smooth at the point where it touches the K-surface and $d\mathbf{e}$ is colinear with $\boldsymbol{\sigma}$, or it has an angular point as in the Tresca condition, when the mean of the possible $d\mathbf{e}$'s has such a direction that $d\mathbf{e}$ is colinear with $\boldsymbol{\sigma}$. The chosen vector $d\mathbf{e}$ has two equal components. In addition it must satisfy the condition

$$de_1 + de_2 + de_3 = 0 \tag{1.5}$$

from which it is easy to obtain expression (1.4).

It should be noted that the value of dW determined from (1.4) remains constant for any arbitrary direction of the vector $\boldsymbol{\sigma}$ under the Mises condition, which has the form here

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 8\ k^2 \tag{1.6}$$

We consider an arbitrary plasticity condition which lies partially inside two Mises circles. We take the point farthest removed from O. At this point the condition

$$|\sigma| > 2K \tag{1.7}$$

is satisfied, Two possibilities may be considered:

a) The flow curve is smooth at this point. Then, by virtue of its maximal distance from the center, its tangent at the point is normal to σ and de is colinear with σ. From the conditions (1.3), (1.5) and (1.7), the inequality

$$dW > 2Kda / \sqrt{(^3/_2)} \tag{1.8}$$

will be satisfied for the point.

b) The flow curve has a singularity at this point. Then, by virtue of its maximal distance in the direction of $d\mathbf{e}$, colinear with $\boldsymbol{\sigma}$, there exist possible means which satisfy condition (1.8). Consequently, if the flow curve falls outside the Mises condition there will exist a value of dW exceeding in value that given by expression (1.4).

We now pass on to the second principle. If the forces which excite the flow are given, then the stresses and the incremental strains are determined as to direction, and the modulus of the stress vector depends upon the plasticity condition. Only a uniform state of stress throughout the body is considered here. Consequently, if (1.2) is investigated for a maximum, the variables will be $|\sigma|$ and the angle between σ and $d\mathbf{e}$, since $d\mathbf{e}$ itself is determined only in direction. Therefore, in this case one may only compare the quantities

$$S = (dW / |d\mathbf{e}|) = |\sigma| \cos \widehat{\sigma\, d\mathbf{e}} \tag{1.9}$$

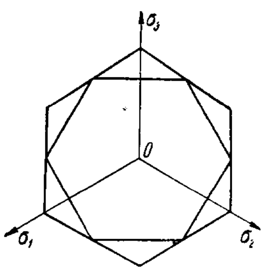

FIG. 2.

When this relation is investigated for a maximum, it is clear that with the forces given, no limit is imposed upon the strains and only the direction of flow is prescribed.

Analogously to the foregoing, the value of

$$S = 2K \tag{1.10}$$

is satisfied at the point A_1 for all plasticity conditions and has a stationary value for the Mises plasticity condition. Evidently S takes on a maximum value at the edges B_1 on the K-surface in the case where $d\mathbf{e}$ and σ are colinear. It is easy to prove that there is a point satisfying the condition

$$S < 2K \qquad (1.11)$$

for any flow curve partially inside the Mises circle

In fact, the Mises plasticity condition will be a lower limit for all plasticity conditions satisfying the relation

$$S \geqslant 2K \qquad (1.12)$$

It is necessary to remark here that $S = 2K$ everywhere on the K-surface except at the edges, where S is reduced from $2K$ to the maximum.

2. The preceding section contained a study of the case in which a unique experimental characteristic provides a limiting value of stress obtained from simple tests in tension or compression. Ivlev showed in his work that other experimental points are not considered, because of imperfection in testing. Ivlev, considering the well-known fact that the majority of tests show better agreement with the Mises condition, attributes this to anisotropy, strain-hardening and similar phenomena. If one agrees with this argument, then it is of interest to note that a similar, purely logical study, with application of experimental principles, may also be made in the case when the experimental data yields a unique value of stress from tests in simple shear. Figure 2 shows the same plane as in Fig. 1 but with the corresponding points given. We shall not furnish a strict proof since it is trivial, but we shall indicate that here also the class of possible plasticity conditions is included between two hexagons. It is especially interesting that here the Tresca condition and the K-surface have changed places.

It is quite evident that the application of extremal principles in this case leads to the same result; i.e. that the K-surface will now be an extremal according to the first principle and the Tresca condition according to the second.

We note in passing that if the Tresca condition is considered from the point of view of the first principle only, as Ivlev did, then it does not exhibit extremal properties in general, but the Mises condition retains all its properties including the stationary property.

We proceed further to investigate the most general case of the given experimental point. Figure 3, analogous to Figs. 1 and 2, shows this point as D_1. It is easy to obtain the boundaries for the class of possible plasticity conditions. In the most general case, all possible plasticity conditions are found between the 12-sided figure and the external broken line, which in itself may not be a flow condition since it is not convex in the most general case.

When investigated, the application of extremal principles leads to the 12-sided figure as the best, and also shows that there is no unique curve. The Mises plasticity condition again conserves its previous properties in this most general case. This result appears to be a consequence of the well-known fact that the Mises plasticity condition is a condition of constant distortion energy.

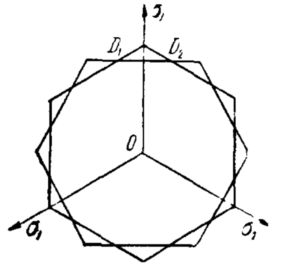

FIG. 3.

It is apparent that the Tresca condition in this most general case does not exhibit extremal properties, since the *K*-surface also does not. And since from the point of view of constructing a simple isotropic theory of ideal plasticity the choice of the given experimental point must not affect the results of the study, then it may be affirmed that in the most general case of the given experimental point only the Mises plasticity condition appears to have an invariant physical characteristic. From this viewpoint Ivlev's conclusion that only the Tresca plasticity condition has physical significance must give rise to doubt. It may be said that the logical construction of a simple plasticity theory on extremal principles leads very successfully to the Mises plasticity condition. In addition, this condition is well supported with a large amount of experimental data.

BIBLIOGRAPHY

1. Ivlev, D.D., K postroeniiu teorii ideal'noi plastichnosti (On the construction of a theory of ideal plasticity). *PMM* Vol. 22, No. 6, 1958.

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